Genus 3 algebraic curves

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Introduction

We describe the basic properties of genus 3 curves, their invariants, automorphism groups, stratification of the moduli space $M_3$, and real models of curves over their field of moduli.

Moduli space $M_3$

Let $M_3$ be the moduli space of genus $g \geq 2$ algebraic curves. A genus $g \geq 2$ curve $C$ has genus $\frac{g-1}{2}$ and a degree $d$ map $C \to P^2$ with ramification structures $(n_1, \ldots, n_m)$.

From the Riemann-Hurwitz formula $\delta = 2g-2$ branch points. Since we can pick a coordinate in $P^2$ by specifying 3 points, then there are $\delta = 3 - \text{parameters}$ left. Hence, $M_3$ in a variety of dimension $\delta = 3$.

$M_3$ is a coarse moduli space of dimension 6. Any curve $C$, $M_3$ is either hyperelliptic or non-hyperelliptic. A point $p \in M_3$ can be described via the invariants of ternary quartics or the invariants of binary octics.

Non-hyperelliptic curves

A non-hyperelliptic curve of genus 3 is a ternary quartic, namely a homogeneous polynomial $F(x, y, z) = t$ of degree 4. From [4] we can take an generic non-hyperelliptic curve of genus 3 has equation $x^3 + y^3 + z^3 + x^2y + y^2z + z^2x = 0$.

Hyperelliptic curves

A curve $C$ is called hyperelliptic if there exists a degree map $2 : C \to P^1$. The set $W$ of branch points of this map is the set of Weierstrass points of $C$. The equation of $C$ is given by $x^2 = \prod (x - a_i)$.

From the Riemann-Hurwitz formula we have exactly 4 + 2 points in $W$. Since we can pick a coordinate in $P^2$ by specifying 3 points, then the equation of $C$ becomes $x^2 = \prod (x - a_i)$.

Non-hyperelliptic curves

We describe both the invariants of ternary quartics and binary octics.

Invariants of ternary quartics

We denote by $V$ the set of ternary quartics. $V$ is a complex vector space of dimension 15. Let $V^J(C)$ be the algebra of complex polynomial functions on $V$ and by $V^J(C)$ the set of homogeneous elements of degree $n$ in $V^J(C)$. Then, $V^J(C)$ is a graded algebra.

The group $\text{GL}(V)$ acts in a natural way on $V$. The invariant subspace under this action is the algebra $A_4$ of projective quartics of plane curves. $A_4$ admits a homogeneous system of parameters of degree 3, 6, 9, 12, 15, 18, 21. We denote these systems of parameters by $I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{21}$, where the $I_n$ are defined as in [Diao 1]. They are homogeneous polynomials of degree $n$, where $n = 3, 6, 9, 12, 15, 18, 21$ and $n$ is coprime with $2$.

There is an algebraic relation $I_3 I_6 = I_{12}$ that restricts the invariants satisfes, compared in [2].

Invariants of binary octics

We denote by $V$ the set of ternary quartics. $V$ is a complex vector space of dimension 15. Let $V^J(C)$ be the algebra of complex polynomial functions on $V$ and by $V^J(C)$ the set of homogeneous elements of degree $n$ in $V^J(C)$. Then, $V^J(C)$ is a graded algebra.

The group $\text{GL}(V)$ acts in a natural way on $V$. The invariant subspace under this action is the algebra $A_4$ of projective octics of plane curves. $A_4$ admits a homogeneous system of parameters of degree 3, 6, 12, 24, 27, 30, 33, 36, 39, 42. We denote these systems of parameters by $I_3, I_6, I_{12}, I_{24}, I_{27}, I_{30}, I_{33}, I_{36}, I_{39}, I_{42}$, where the $I_n$ are defined as in [Diao 1]. They are homogeneous polynomials of degree $n$, where $n = 3, 6, 12, 24, 27, 30, 33, 36, 39, 42$ and $n$ is coprime with $2$.

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In the following table we display all possible automorphism groups of hyperelliptic curves, the dimensions of the corresponding loci, and the parametric equation of the curve. Groups are identified by their GAP identifier.

Automorphisms

<table>
<thead>
<tr>
<th>Aut $\phi$</th>
<th>$\phi$</th>
<th>equation $y = f(x)$</th>
<th>Id</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{S}_4(3)$</td>
<td>$\phi_1$</td>
<td>$x^3 + y^3 + z^3 + x^2y + y^2z + z^2x = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{S}_4(6)$</td>
<td>$\phi_2$</td>
<td>$x^3 + y^3 + z^3 + x^2y + y^2z + z^2x = 0$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{S}_4(9)$</td>
<td>$\phi_3$</td>
<td>$x^3 + y^3 + z^3 + x^2y + y^2z + z^2x = 0$</td>
<td>3</td>
</tr>
</tbody>
</table>

The groups for non-hyperelliptic curves are given below.

<table>
<thead>
<tr>
<th>$G$</th>
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<th>Group ID</th>
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Each of these cases is an irreducible locus in $M_3$.

Dihedral invariants

Let $X$ be a genus 3 hyperelliptic curve, $X$, defined over $k$, has an non-hyperelliptic involution. This corresponds to the case 2 in the Table of hyperelliptic curves. Then, the following automorphism $\phi: X \to X$ is an automorphism of $X$.

<table>
<thead>
<tr>
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Theorem: $\text{S}_4(3)$ is an automorphism group $\phi$.

Hence, $\phi_1, \phi_2, \phi_3$ can be expressed as rational functions in $t_1, \ldots, t_6$. The following theorem determines relations among $\phi_1, \phi_2, \phi_3$ for each group $G$ such that $V_1 \to G$.

Theorem: Let $X$ be a curve in $S \sim M_3$. Then, one of the following occurs:

1. $\phi: X \to X$ is an automorphism of $X$.
2. $\phi: X \to X$ is an automorphism of $X$.
3. $\phi: X \to X$ is an automorphism of $X$.
4. $\phi: X \to X$ is an automorphism of $X$.
5. $\phi: X \to X$ is an automorphism of $X$.

Equations over the field of moduli

If $\text{Aut}(\phi)(X)$ then $X$ is isomorphic to a curve with equation $y^3 = (x^2 + a_1)x^2 + (x^2 + a_2)x + x^2 + a_3$.

Stratification of the moduli space $M_3$

We can determine the inclusions among the loci given in Tables 1 and 2. For the hyperelliptic moduli such loci are described algebraically by relations among $e_i, a_i, c_i$ or equivalently $\alpha_i$.

References